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Topical Results on Lattice Chiral Fermions in the CFA*

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We report new results on the lattice regularization of the chiral Schwinger model and the chiral U(1) model in four dimensions in the CFA.

1. INTRODUCTION

The continuum fermion approach (CFA) to regularizing chiral fermions appears to be a promising method. For a summary of results obtained so far see [1]. The basic idea of the approach is the following. We start from a lattice with spacing a. We call this the original lattice. On this lattice the simulations are done. Next we construct a finer lattice with lattice spacing a_f, using a suitable interpolation of the gauge field [2]. On this lattice we formulate the fermions. The action for a single fermion of chirality ϵ_α is taken to be

$$\begin{aligned} S_{\epsilon_\alpha} &= \bar{\psi}\mathcal{D}_{\epsilon_\alpha}\psi \\ &\equiv \frac{1}{2}\bar{\psi}\{\gamma_\mu(D_\mu^{\epsilon_\alpha +} + D_\mu^{\epsilon_\alpha -}) \\ &- \frac{r}{2}a_f D_\mu^{\epsilon_\alpha +} D_\mu^{\epsilon_\alpha -}\}\psi, \end{aligned} \tag{1}$$

where, restricting ourselves to gauge group U(1),

$$\begin{aligned} (D_\mu^{\epsilon_\alpha \pm}\psi)(n) &= \pm\frac{1}{a_f}\{[P_{-\epsilon_\alpha} + P_{\epsilon_\alpha}(U_{\pm\mu}^f(n))^{e_\alpha}] \\ &\quad \times \psi(n \pm \hat{\mu}) - \psi(n)\}, \end{aligned} \tag{2}$$

with $U_{\pm\mu}^f$ being the link variable on the fine lattice, e_α the fractional charge, and $P_{\epsilon_\alpha} = (1 + \epsilon_\alpha\gamma_5)/2$. The effective action is then computed from (1) in the limit $a_f \to 0$, where a is kept fixed. Hence the name continuum fermions.

A particular feature of this action – or better of the Wilson term we have chosen – is that the Wilson-Dirac operator $\mathcal{D}_{\epsilon_\alpha}$ fulfills the Ginsparg-Wilson relation [3,4]:

$$\gamma_5\mathcal{D}_{\epsilon_\alpha} + \mathcal{D}_{\epsilon_\alpha}\gamma_5 = r\, a_f \mathcal{D}_{\epsilon_\alpha}\gamma_5\mathcal{D}_{\epsilon_\alpha} + O(a_f^2). \tag{3}$$

This is not a great surprise. The motivation behind our construction was to find an action which obeys the index theorem – what, in fact, it does as we shall see below. The Ginsparg-Wilson relation (3) does, however, not guarantee that the theory is invariant under chiral gauge transformations.

Indeed, the resulting effective action, W_{ϵ_α}, is not gauge invariant, even in the limit $a_f \to 0$. But there exists a local, purely bosonic counterterm C, so that

$$\widehat{W}_{\epsilon_\alpha} = \lim_{a_f \to 0} W_{\epsilon_\alpha}^\Sigma,\ W_{\epsilon_\alpha}^\Sigma = W_{\epsilon_\alpha} + C \tag{4}$$

is invariant under chiral gauge transformations. For the imaginary part this is only true in the anomaly-free model. The counterterm can be – and has been – computed in perturbation theory.

A further important feature of the action (1) is that it has a shift symmetry which makes sure that the ungauged fermion with chirality $-\epsilon_\alpha$ decouples.

In the chiral Schwinger model we found for the topologically trivial sector of the theory

$$\widehat{W}_{\epsilon_\alpha} = \frac{1}{2}(W_V + W_0) + \mathrm{i}\,\mathrm{Im}W_{\epsilon_\alpha}, \tag{5}$$

where W_V and W_0 are the effective actions of the corresponding vector model and the free theory,

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respectively. The imaginary part of the effective action turned out to be given by the harmonic part of the gauge field, i.e the toron field, alone, and it could be computed analytically from the gauge field we started with. Thus we arrived at an action which can be simulated relatively easily on the original lattice. We believe to find a similar result in more realistic models in four dimensions.

After so much of introduction, let us now come to our new results.

2. CHIRAL SCHWINGER MODEL

Our first results concern the chiral Schwinger model. Here we want to test whether the index theorem is fulfilled. This is a non-trivial task because any topologically non-trivial gauge field involves at least one singular plaquette. Some authors have argued that the CFA would fail this test. This would be true had we employed the standard gauged or ungauged Wilson terms. Furthermore, we will investigate whether our approach reproduces the correct anomaly.

In two dimensions the index theorem says that in a background gauge field configuration of topological charge Q we should find exactly

$$n_{\epsilon_\alpha} = |Q|\,\theta(\epsilon_\alpha Q) \tag{6}$$

zero modes of chirality ϵ_α. For a configuration of, e.g., charge $Q = +1$ this would mean $n_+ = 1$ and $n_- = 0$.

To leading order in a_f the Wilson-Dirac operator can be written

$$\mathcal{D}_{\epsilon_\alpha} = \not{D}^{\epsilon_\alpha} - a_f \frac{r}{2}\, \not{D}^{-\epsilon_\alpha}\, \not{D}^{\epsilon_\alpha} + O(a_f^2), \tag{7}$$

where $\not{D}^{\epsilon_\alpha}$ is the average of forward and backward derivatives. Being a finite matix, the Dirac operator $\not{D}^{\epsilon_\alpha} \equiv -\not{D}^{-\epsilon_\alpha\,\dagger}$ has the same number of zero modes as the corresponding vector operator, namely $n_{\epsilon_\alpha} + n_{-\epsilon_\alpha} = |Q|$, thus violating the index theorem. But the situation is different for the Wilson term $\not{D}^{-\epsilon_\alpha}\,\not{D}^{\epsilon_\alpha} \equiv (\not{D}^{-\epsilon_\alpha}\,\not{D}^{\epsilon_\alpha})^\dagger$. It has n_{ϵ_α} zero modes of chirality ϵ_α and none of chirality $-\epsilon_\alpha$, exactly as required by the index theorem. For small, but finite a_f the zero modes approach the value

$$\left(\frac{a_f}{a}\right)^2 \frac{\pi|Q|}{L^2}, \tag{8}$$

where L is the size of the original lattice.

We consider two configurations of charge Q. We denote the link variables on the original lattice by $U_\mu(s) = \exp(\mathrm{i}\theta_\mu(s))$, $-\pi < \theta_\mu(s) \le \pi$, where s_μ are the lattice points on the original lattice. The first configuration is [5] (mod 2π)

$$\begin{aligned} \theta_1(s) &= F\, s_2 - \bar{\theta}_1, \\ \theta_2(s) &= \begin{cases} -\bar{\theta}_2, & s_2 = 1, \ldots, L-1, \\ FL\, s_1 - \bar{\theta}_2, & s_2 = L, \end{cases} \end{aligned} \tag{9}$$

where $F = 2\pi Q/L^2$, $\bar{\theta}_1 = \pi(L-1)/L^2$ and $\bar{\theta}_2 = \pi/L^2$. The second configuration is

$$\theta_\mu(s) = 2\pi Q \epsilon_{\mu\nu} \partial_\nu^- G(s-\bar{s}),\ \bar{s} = (L/2, L), \tag{10}$$

where G is the inverse lattice Laplacian. Both configurations have constant field strength F, zero toron field, and for $|Q| = 1$ they have one singular plaquette at $s = (L/2, L)$. The two configurations are related by a periodic (topologically trivial) gauge transformation. In the following we shall take $L = 6$, and we shall use the interpolation given in [2].

Both configurations give the same value for the effective action (4). As the anomaly-free model we took $\epsilon_\alpha e_\alpha = -1, -1, -1, -1, 2$. This indicates that we have gauge invariance in the background of singular gauge fields as well. For $Q = 1$ we find exactly one zero mode with chirality $+$, and none with chirality $-$. For $Q = 2$ we find two zero modes with chirality $+$, and none with chirality $-$. For negative charges we find the same result but with $+$ and $-$ interchanged, in agreement with the index theorem. The values of the zero modes are in good agreement with the analytical result (8).

A further requirement of the method is that it reproduces the correct anomaly. In the chiral Schwinger model the anomaly condition reads

$$\frac{\delta W_{\epsilon_\alpha}}{\delta h(s)} = \mathrm{i}\epsilon_\alpha e_\alpha^2 \frac{1}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}(s), \tag{11}$$

where $h(s)$ is a gauge transformation. We consider configuration (10) with $Q = -1$ and $\bar{s} = (L/2, L/2)$ now. The r.h.s. of (11) is known analytically. The l.h.s. is computed for $a/a_f = 5$, i.e. on the 30^2 lattice. In Fig. 1 we show the result for both sides separately for $\epsilon_\alpha e_\alpha = -1$ and 2, as well

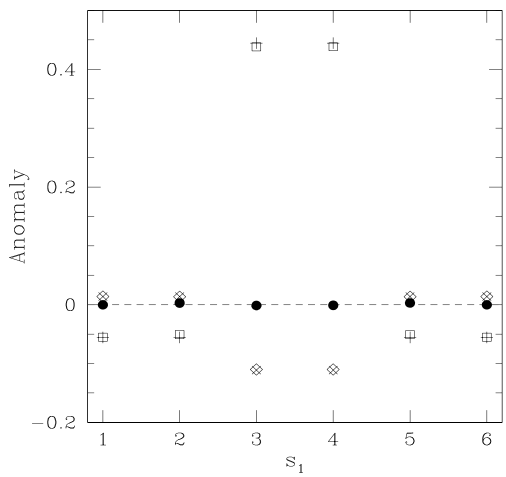

Figure 1. The anomaly as a function of s_1 for $s_2 = 4$. The l.h.s. (r.h.s.) of (11) is marked by $\Box$ (+) for $\epsilon_\alpha e_\alpha = -1$, and by $\Diamond$ ($\times$) for $\epsilon_\alpha e_\alpha = 2$. The result for the anomaly-free model is denoted by $\bullet$.

as for the anomaly-free model. We find excellent agreement between our numerical results and the theoretical expectations (i.e. the l.h.s. versus the r.h.s.). Moreover, we see that the anomaly cancels in the anomaly-free model. We find similar results for configuration (9).

3. CHIRAL U(1) MODEL IN 4-D

Our other new results are on the chiral U(1) model in four dimensions. This model is simpler than the chiral Schwinger model because it has no topological charge and zero modes to worry about.

The first question is again whether the theory exists at all, i.e. can be made finite by the addition of local counterterms. We find three possible counterterms:

$$\begin{aligned} C_2 &= c_2 \sum \left(\sum_\mu A_\mu^2 \right), \\ C_4 &= c_4 \sum \left(\sum_\mu A_\mu^2 \right)^2, \\ C_\partial &= c_\partial \sum \left(\sum_\mu \partial_\mu A_\mu \right)^2. \end{aligned} \tag{12}$$

As before, the coefficients, c_2, c_4 and c_∂, can be computed in lattice perturbation theory. The result so far is

$$c_2 = 0.03717\, a_f^{-2},\ c_4 = 0.00052. \tag{13}$$

We cannot quote any number for c_∂ yet. Numerically we find that gauge invariance is, within our present accuracy, already restored by the counterterms C_2 and C_4, if we use the perturbative values (13) for the coefficients. This indicates that c_∂ is very small, or zero. As an upper bound we can quote $c_\partial < 10^{-4}$.

What distinguishes the chiral theory from the corresponding vector theory is the imaginary part, $\mathrm{Im}\, W_{\epsilon_\alpha}$, of the effective action. In two dimensions this was entirely determined by the toron field contribution. In four dimensions, however, we find that it is zero for toron field configurations. This leaves the interesting possibility that the imaginary part of the effective action is generally zero in this model.

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